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A STUDY OF THE INSTABILITY OF LIQUID JETS AND COMPARISON WITH TOMOTIKA'S ANALYSIS

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Abstract—A novel dispersion equation describing the instability of low velocity, liquid jets is derived using an integro-differential approach. The explicit form of the derived equation enables much easier prediction of the most unstable wavenumber and disturbance growth rate than does Tomotika's implicit, complex dispersion relationship. Good agreement is observed with all limiting solutions to Tomotika's relationship and with numerical solutions to non-limiting cases. The influences of density ratio, viscosity ratio and Ohnesorge number on liquid jet instability are examined.

Key Words: liquid jet, capillary instability

1. INTRODUCTION

The injection of a liquid into another fluid has many industrial applications. Due to capillary instability, the injected liquid (the jet) becomes unstable and breaks up into droplets. According to Plateau (1873), capillary instability arises as a result of interfacial tension whenever the wavelength of the surface disturbance exceeds the circumference of the cylindrical liquid jet; i.e. capillary instability is caused by long disturbance waves. Rayleigh (1878) used rigorous theoretical analysis to show that, from an initially small disturbance, a number of unstable waves may form on the jet surface; the wave that causes the jet to breakup, the "most unstable wave", is that which has the maximum growth rate in amplitude. Rayleigh (1878, 1892a, b) applied this maximum instability theory to liquid-into-gas and gas-into-liquid systems. For viscous liquid jets, Rayleigh (1892a) obtained only a limiting solution. Weber (1931) derived a dispersion equation for a viscous liquid jet issuing into gas. Weber found that for viscous liquid jets, the most unstable wavelength is longer than that predicted by Rayleigh for inviscid liquid-into-gas systems (Rayleigh 1878). Following Rayleigh's approach and modeling the flows for both jet and ambient fluids as Stokes flows, Tomotika (1935) obtained a dispersion equation for a viscous liquid jet in another viscous liquid. Tomotika found that the instability of the jet is strongly influenced by the ratios of the viscosities and densities of the jet and ambient fluids, and the Ohnesorge number, a dimensionless parameter representing the ratio of viscous:interfacial-tension forces.

Tomotika's equation is a general dispersion equation applicable to jets with low velocities. It contains more information than any dispersion equation obtained previously for the same applications; however, it is an implicit equation in complex form, and therefore is difficult to use in determining the most unstable wave and related information. Several attempts have been made to solve Tomotika's equation (Meister 1966; Meister & Scheele 1967; Takahashi & Kitamura 1971; Lee 1972; Lee & Flumerfelt 1981). To date, only a few limiting solutions have been obtained (Meister & Scheele 1967; Lee & Flumerfelt 1981) and, unfortunately, those limiting solutions do not apply to many liquid-into-liquid and liquid-into-gas systems of practical importance. In addition, the lack of clearly defined bounds of validity has often led to inappropriate use of those limiting solutions. An equivalent generalized explicit dispersion equation would eliminate such difficulties and facilitate analyzing the influences of the jet and ambient fluid properties on instability. This study seeks to develop an explicit dispersion equation under the same limits of analysis, principally Stokes flows for both phases, that apply to Tomotika's implicit equation.

2. GOVERNING EQUATIONS

The flow system under consideration consists of an infinitely long, low-velocity, viscous, cylindrical liquid jet issuing into another fluid (liquid or gas). Both the jet and ambient fluids are assumed to be Newtonian and incompressible. Heat and mass transfer effects are neglected. Following Rayleigh (1892a) and Tomotika (1935), the motions of both the jet and ambient fluids are modeled as Stokes flows. Conservation of energy for the jet gives

$$\int_V \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u_i^2\right) \mathrm{d}V + \int_V \tau_{ij}\epsilon_{ij}\,\mathrm{d}V - \int_S \tau_{ij}u_i n_j\,\mathrm{d}S = 0, \qquad [1]$$

where V is the volume of the jet, t is time, ρ is density, u_i is the velocity component in the ith direction, τ_{ij} is the stress tensor, ϵ_{ij} is the strain-rate tensor, S is the area of the jet surface and n_j is the projection of the unit outward normal to the jet. The first term in [1] represents the rate of increase in kinetic energy; the second term, the rate of energy dissipation; and the third term, the rate at which work is performed on the jet. The momentum and continuity equations that govern the motion of the ambient fluid are

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} = -\frac{1}{\hat{\rho}}\nabla\hat{p} + \frac{\hat{\mu}}{\hat{\rho}}\nabla^2\hat{\mathbf{u}} \qquad [2]$$

and

$$\nabla\cdot\hat{\mathbf{u}} = 0, \qquad [3]$$

where the caret signifies properties of the ambient fluid, p is the pressure and μ is the viscosity. The jet/ambient fluid interfacial conditions are:

kinematic conditions,

$$u_{rs} = \hat{u}_{rs}, \quad u_{zs} = \hat{u}_{zs}; \qquad [4]$$

and

dynamic conditions,

$$\tau_{rrs} = -\hat{p} + 2\hat{\mu}\left(\frac{\partial \hat{u}_r}{\partial r}\right)_s - \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad [5]$$

and

$$\tau_{rzs} = \hat{\mu}\left(\frac{\partial \hat{u}_z}{\partial r} + \frac{\partial \hat{u}_r}{\partial z}\right)_s, \qquad [6]$$

where the subscript s denotes the properties at the interface, σ is the interfacial tension and r_1 and r_2 are the principal radii of curvature of the interface.

3. INSTABILITY ANALYSIS

It is assumed that the jet surface is initially perturbed infinitesimally; the radius of the disturbed jet, r_s, can be expressed as (Plateau 1873; Rayleigh 1878; Weber 1931; Tomotika 1936):

$$r_s = a + \alpha(t)\cos kz, \qquad [7]$$

where a is the undisturbed jet radius, α is the disturbance amplitude and k is the wavenumber. For low velocity conditions, the jet may be modeled as a one-dimensional Cosserat continuum (Bogy 1978, 1979). Then, continuity leads to (Levich 1962; Lee 1974; Bogy 1978):

$$\frac{\partial A}{\partial t} + \frac{\partial (Au_z)}{\partial z} = 0, \qquad [8]$$

where $A = \pi r_s^2$ is the jet cross-sectional area. Equation [8] yields the following approximate expression:

$$\frac{\partial u_z}{\partial z} \approx -\frac{2}{a}\dot{\alpha}\cos kz,$$

where $\dot{\alpha} = \mathrm{d}\alpha/\mathrm{d}t$. Thus,

$$u_z = -\frac{2\dot{\alpha}}{ka}\sin kz. \tag{9}$$

Continuity for the jet fluid yields

$$u_r = \frac{r}{a}\dot{\alpha}\cos kz. \tag{10}$$

The above expressions for u_r and u_z can be used to recast the first and second terms in [1]:

$$\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right)\mathrm{d}V = \frac{\pi^2 m a^3 \rho}{2\eta^3}(8+\eta^2)\dot{\alpha}\ddot{\alpha} \tag{11}$$

and

$$\int_V \tau_{ij}\epsilon_{ij}\,\mathrm{d}V = \frac{\pi^2 m a \mu}{2\eta}(24+\eta^2)\dot{\alpha}^2, \tag{12}$$

where $\ddot{\alpha} = \mathrm{d}^2\alpha/\mathrm{d}t^2$ and $\eta = ka$. (The integrations are performed between $z = 0$ and $2\pi m/k$, over m integral number of waves.) The work term in [1] can be expressed as

$$\int_S \tau_{ij}u_i n_j\,\mathrm{d}S = \int_S \tau_{rr}u_r n_r\,\mathrm{d}S + \int_S \tau_{rz}u_z n_r\,\mathrm{d}S.$$

Applying the interfacial conditions,

$$\int_S \tau_{ij}u_i n_j\,\mathrm{d}S = -\int_S \hat{p}u_r\,\mathrm{d}S + \int_S 2\hat{\mu}\left(\frac{\partial \hat{u}_r}{\partial r}\right)_{\mathrm{s}} u_r\,\mathrm{d}S - \int_S \sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right)u_r\,\mathrm{d}S + \int_S \hat{\mu}\left(\frac{\partial \hat{u}_z}{\partial r}+\frac{\partial \hat{u}_r}{\partial z}\right)_{\mathrm{s}} u_z\,\mathrm{d}S. \tag{13}$$

The interfacial tension term in [13] may be written as

$$\sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = \sigma\left\{\frac{1}{r_{\mathrm{s}}} + \frac{\dfrac{\partial^2 r_{\mathrm{s}}}{\partial z^2}}{\left[1+\left(\dfrac{\partial r_{\mathrm{s}}}{\partial z}\right)^2\right]^{3/2}}\right\}$$

$$\approx \frac{\sigma}{a} - \frac{\alpha\sigma}{a^2}(1-\eta^2)\cos kz,$$

where σ/a represents the undisturbed component which is eliminated in instability analyses (Rayleigh 1878). Following Rayleigh, the surface integral in [13] containing the interfacial tension term is modified to

$$\int_S\left[\sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-\frac{\sigma}{a}\right]u_r\,\mathrm{d}S = -\frac{2\pi^2 m\sigma}{\eta}(1-\eta^2)\alpha\dot{\alpha}. \tag{14}$$

Equation [2] can be rewritten as

$$\frac{\partial \hat{u}_z}{\partial t} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial z} + \frac{\hat{\mu}}{\hat{\rho}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{u}_z}{\partial r}\right)+\frac{\partial^2 \hat{u}_z}{\partial z^2}\right] \tag{15}$$

and

$$\frac{\partial \hat{u}_r}{\partial t} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial r} + \frac{\hat{\mu}}{\hat{\rho}}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}_r)\right)+\frac{\partial^2 \hat{u}_r}{\partial z^2}\right]. \tag{16}$$

For capillary instability, only disturbance waves having wavelengths longer than the circumference of the jet are of interest. Based on Rayleigh's maximum instability theory (Rayleigh 1878), it can be assumed that $\lambda \gg a$. Equations [9], [10] and [4] suggest that $\hat{u}_{zs} \sim (\lambda/a)\hat{u}_{rs}$. Thus, $\hat{u}_z \sim (\lambda/a)\hat{u}_r$ and $\hat{u}_z \gg \hat{u}_r$. Selecting λ and a as characteristic lengths, [15] and [16] can be rewritten in dimensionless form as

$$\frac{\partial \hat{u}_z^*}{\partial t^*} = -\frac{\partial \hat{p}^*}{\partial z^*} + \frac{\lambda}{a}\frac{1}{\mathrm{Re}}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \hat{u}_z^*}{\partial r^*}\right)+\left(\frac{a}{\lambda}\right)^2\frac{\partial^2 \hat{u}_z^*}{\partial z^{*2}}\right] \tag{17}$$

and

$$\frac{a}{\lambda}\frac{\partial \hat{u}_r^*}{\partial t^*} = -\frac{\partial \hat{p}^*}{\partial r^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*\hat{u}_r^*)\right) + \left(\frac{a}{\lambda}\right)^2 \frac{\partial^2 \hat{u}_r^*}{\partial z^{*2}}\right], \quad [18]$$

where $\hat{u}_z^* \equiv \hat{u}_z/U$, $\hat{u}_r^* \equiv \hat{u}_r/U$, $t^* \equiv tU/\lambda$, $r^* \equiv r/a$, $z^* \equiv z/\lambda$ and $\hat{p}^* \equiv \hat{p}/\hat{\rho}U^2$ are dimensionless variables; $\mathrm{Re} \equiv Ua\hat{\rho}/\hat{\mu}$ is the Reynolds number and U is the unperturbed jet velocity. Thus, $\lambda/a \gg 1$, $\hat{u}_z^* \sim 1$, $\hat{u}_r^* \ll 1$ and $z^* \sim 1$. The order of magnitude of r^* depends on the region of interest; two separate regions are considered.

Region 1: $r_s < r < \lambda$

In this region, it is assumed that $r^* \sim 1$. An order-of-magnitude analysis reduces [17] and [18] to

$$\frac{\partial \hat{u}_z^*}{\partial t^*} = -\frac{\partial \hat{p}^*}{\partial z^*} + \frac{\lambda}{a}\frac{1}{\mathrm{Re}}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \hat{u}_z^*}{\partial r^*}\right) \quad [19]$$

and

$$\frac{\partial \hat{p}^*}{\partial r^*} \rightarrow 0. \quad [20]$$

Region 2: $r > \lambda$

In this region, $\lambda/(ar^*) = \lambda/r < 1$ and $\partial \hat{u}_z^*/\partial r^* \rightarrow 0$. The diffusion term in region 2 is negligible in comparison with the equivalent term in region 1; thus,

$$\frac{\partial \hat{u}_z^*}{\partial t^*} = -\frac{\partial \hat{p}^*}{\partial z^*},$$

i.e.

$$\frac{\partial \hat{u}_z}{\partial t} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial z}. \quad [21]$$

Modeling the flow in region 2 as potential flow, $\hat{u}_z$ is rewritten as $\hat{u}_z = \partial\varphi/\partial z$. The velocity potential, φ, must satisfy Laplace's equation

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

Equation [21] becomes

$$\frac{\partial^2\varphi}{\partial t\,\partial z} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial z};$$

thus,

$$\frac{\partial\varphi}{\partial t} = -\frac{\hat{p}}{\hat{\rho}}. \quad [22]$$

Solving for φ and substituting into [22] yields

$$\frac{\hat{p}}{\hat{\rho}} = \frac{\ddot{\alpha}}{k}\frac{K_0(kr)}{K_1(\eta)}\cos kz, \quad [23]$$

where K_0 and K_1 are the zeroth- and first-order modified Bessel functions of the second kind. The pressure term in [13] becomes

$$\int_S \hat{p}u_r\,\mathrm{d}S = \frac{2\pi^2 m\hat{\rho}a^3}{\eta^2}\frac{K_0(\eta)}{K_1(\eta)}\dot{\alpha}\ddot{\alpha}. \quad [24]$$

Equation [23], subject to [21], gives

$$\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial z} = -\ddot{\alpha}\frac{K_0(kr)}{K_1(\eta)}\sin kz$$
$$= \frac{\eta}{2}\frac{\partial}{\partial t}\left(-\frac{2\dot{\alpha}\sin kz}{\eta}\right)\frac{K_0(kr)}{K_1(\eta)}$$
$$= \frac{\eta}{2}\frac{\partial \hat{u}_{zs}}{\partial t}\frac{K_0(kr)}{K_1(\eta)}.$$

Equation [19] becomes

$$\frac{\partial \hat{u}_z^*}{\partial t^*} = -\frac{\eta}{2}\frac{\partial \hat{u}_{zs}^*}{\partial t^*}\frac{K_0(kr)}{K_1(\eta)} + \frac{\lambda}{a}\frac{1}{\mathrm{Re}}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \hat{u}_z^*}{\partial r^*}\right). \qquad [25]$$

The objective at this stage is to solve the surface integrals in [13] (those containing viscosity); hence, in [25], attention is focused on the interfacial zone, i.e. in the region $r_s \leqslant r < r_s + a$. In that region, $r^* \sim 1$. For Stokes flow, $\mathrm{Re} < 1$. Comparing the orders of magnitude of the terms in [25] gives

$$\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \hat{u}_z^*}{\partial r^*}\right) = 0,$$

i.e.

$$\frac{\partial}{\partial r}\left(r\frac{\partial \hat{u}_z}{\partial r}\right) = 0. \qquad [26]$$

The solution to [26] is

$$\hat{u}_z = C_1\left[\ln\left(\frac{a}{r}\right) + C_2\right], \qquad [27]$$

where C_1 and C_2 are functions of t and z. In the region $r > \lambda$,

$$\hat{u}_z = \frac{\partial \varphi}{\partial z} = \dot{\alpha}\sin kz\frac{K_0(kr)}{K_1(\eta)}.$$

The assumption $\lambda \gg a$ implies that $\eta < 1$, in which case, $K_1(\eta) \sim 1/\eta$. Therefore,

$$\hat{u}_z \approx \dot{\alpha}\eta \sin kz K_0(kr). \qquad [28]$$

Also (based on [4] and [9]),

$$\hat{u}_{zs} = -\frac{2\dot{\alpha}}{\eta}\sin kz. \qquad [29]$$

Equations [28] and [29] differ only in their functional dependence on r; thus,

$$C_1 = -\dot{\alpha}\eta\sin kz \quad \text{and} \quad C_2 = \frac{2}{\eta^2}.$$

Hence,

$$\hat{u}_z = -\dot{\alpha}\eta\sin kz\left[\ln\left(\frac{a}{r}\right) + \frac{2}{\eta^2}\right]. \qquad [30]$$

(Note, [30] is valid only near the interface; although the velocity distribution in the region $r_s + a < r < \lambda$ is not known, it is not relevant to solving those surface integrals in [13] containing viscosity.) From [30],

$$\left(\frac{\partial \hat{u}_z}{\partial r}\right)_s = \frac{\dot{\alpha}\eta\sin kz}{r_s}. \qquad [31]$$

Based on the preceding,

$$\int_S \hat{\mu}\left(\frac{\partial \hat{u}_z}{\partial r}+\frac{\partial \hat{u}_r}{\partial z}\right)_s u_z \, \mathrm{d}S + \int_S 2\hat{\mu}\left(\frac{\partial \hat{u}_r}{\partial r}\right)_s u_r \, \mathrm{d}S = \int_S \hat{\mu}\left(\frac{\partial \hat{u}_z}{\partial r}u_z+\frac{\partial \hat{u}_r}{\partial z}u_z+2\frac{\partial \hat{u}_r}{\partial r}u_r\right)_s \mathrm{d}S$$
$$\approx \int_S \hat{\mu}\left(\frac{\partial \hat{u}_z}{\partial r}\right)_s u_z \, \mathrm{d}S = \frac{-4\pi^2 ma\hat{\mu}\dot{\alpha}^2}{\eta}. \qquad [32]$$

All integrals in [13] have been recast as derivatives. Substituting [14], [24] and [32] into [13] and combining with [11] and [12] in [1], and rearranging

$$\left[1+\frac{1}{2}\frac{\hat{\rho}}{\rho}\frac{\eta}{1+\frac{\eta^2}{8}}\frac{K_0(\eta)}{K_1(\eta)}\right]\ddot{\alpha}+\left[3\frac{\mu}{\rho a^2}\eta^2\frac{1+\frac{\eta^2}{24}}{1+\frac{\eta^2}{8}}+\frac{\hat{\mu}}{\rho a^2}\frac{\eta^2}{1+\frac{\eta^2}{8}}\right]\dot{\alpha}=\frac{1}{2}\frac{\sigma}{\rho a^3}\eta^2\frac{1-\eta^2}{1+\frac{\eta^2}{8}}\alpha; \qquad [33]$$

$\alpha = \alpha_0 \mathrm{e}^{\omega t}$, where α_0 is the initial disturbance and ω is the growth rate, satisfies [33]. The terms $\eta^2/8$ and $\eta^2/24$ in [33] can be neglected based on the condition $\lambda \gg a$. Substituting α into [33] and rearranging yields the following dimensionless dispersion equation:

$$\left[1+\frac{1}{2}\frac{\hat{\rho}}{\rho}\eta\frac{K_0(\eta)}{K_1(\eta)}\right]\beta^2+2Z\left(3+\frac{\hat{\mu}}{\mu}\right)\eta^2\beta=\eta^2(1-\eta^2), \qquad [34]$$

where $\beta \equiv \omega(2\rho a^3/\sigma)^{1/2}$ is the dimensionless disturbance growth rate and $Z \equiv \mu(2a\rho\sigma)^{-1/2}$ is the Ohnesorge number based on the jet fluid. The jet is unstable whenever $\beta > 0$ ($\omega > 0$). The dispersion equation [34] is valid for either liquid-into-liquid or liquid-into-gas systems, as long as the motions of both the jet and ambient fluids can be modeled as Stokes flows. If the higher-order terms in [33] are retained and if ambient effects are neglected, [33] reduces to the solution obtained by Bogy for liquid-into-gas jets [i.e. [19] in Bogy (1978)].

The conditions assumed in developing [34] are exactly the same as those applied in Tomotika's analysis (Tomotika 1935). In that analysis, as in many other studies (e.g. Rayleigh 1892a; Weber 1931), the dispersion equation is derived by applying the appropriate interfacial conditions. By contrast, in this study, the dispersion equation is obtained by conducting an energy balance on the jet. An advantage of the present approach is that, by modeling the jet as a one-dimensional Cosserat continuum, the jet velocity can be determined easily and the ambient velocity can be solved using boundary-layer techniques; and, as discussed below, the resultant solution, [34], is much simpler than Tomotika's implicit dispersion equation.

4. COMPARISON WITH TOMOTIKA'S ANALYSIS

Tomotika's dispersion equation (Tomotika 1935) can be expressed as

$$\det([a_{ij}]) = 0, \qquad [35]$$

where $[a_{ij}]$ is a 4×4 matrix. All elements in the above matrix contain Bessel functions of η, η_1 or $\hat{\eta}$, where

$$\eta_1^2 \equiv \eta^2+\frac{\beta}{2Z}, \quad \hat{\eta}^2 \equiv \eta^2+\frac{\beta}{2Z}\frac{\hat{\rho}}{\rho}\frac{\mu}{\hat{\mu}}.$$

In concept, the dimensionless amplitude growth rate may be determined from [35] as

$$\beta = \beta\left(\eta, \eta_1, \hat{\eta}, \frac{\hat{\mu}}{\mu}, \frac{\hat{\rho}}{\rho}, Z\right).$$

Solutions to limiting cases of Tomotika's equation are discussed in detail by Meister (1966), Meister & Scheele (1967), Lee (1972), and Lee & Flumerfelt (1981). Lee & Flumerfelt (1981) presented 11 limiting cases; however, those actually can be reduced to 7 distinct limiting cases. The following discussion compares the results of the present investigation, [34], with the limiting solutions to Tomotika's equation, [35], as obtained by Meister & Scheele (1967) and Tomotika (1935).

Case 1: an inviscid liquid jet in a gas

For the case of an inviscid liquid jet in an ambient gas, [35] reduces to (Meister & Scheele 1967):

$$\beta^2 = 2\eta(1-\eta^2)\frac{I_0(\eta)}{I_1(\eta)}, \qquad [36]$$

which is identical to the classical result obtained by Rayleigh (1878). Rayleigh determined that the most unstable wavenumber is $\eta_m = 0.697$. Rewriting [34],

$$\left[1+\frac{1}{2}\frac{\hat{\rho}}{\rho}\eta\frac{K_0(\eta)}{K_1(\eta)}\right]\beta^2 + \frac{2}{(2a\rho\sigma)^{1/2}}(3\mu+\hat{\mu})\eta^2\beta = \eta^2(1-\eta^2), \qquad [37]$$

and substituting the appropriate conditions for this specific case, viz. $\hat{\rho}/\rho = 0$, $\mu = 0$ and $\hat{\mu} = 0$, into [37] gives

$$\beta^2 = \eta^2(1-\eta^2). \qquad [38]$$

Since $I_1(\eta)/I_0(\eta) \sim \eta/2$ for $\eta < 1$, [38] is essentially the same as the classical result, [36]. Equation [38] yields a most unstable wavenumber of $\eta_m = 0.707$.

Case 2: a gas jet in an inviscid liquid

For the case of a gas jet in an inviscid liquid, [35] reduces to (Meister & Scheele 1967):

$$\beta^2 = 2\frac{\rho}{\hat{\rho}}\eta(1-\eta^2)\frac{K_1(\eta)}{K_0(\eta)}, \qquad [39]$$

which is identical to the classical result obtained by Rayleigh (1892b). Rayleigh determined that the most unstable wavenumber for this case is $\eta_m = 0.485$. Substituting the appropriate conditions for this case, $\hat{\rho}/\rho \gg 1$, $\hat{\mu}/\mu \to 0$ and $Z \to 0$, into [34] of the present study yields

$$\frac{1}{2}\frac{\hat{\rho}}{\rho}\eta\frac{K_0(\eta)}{K_1(\eta)}\beta^2 = \eta^2(1-\eta^2), \qquad [40]$$

which is identical to [39].

Case 3: an inviscid liquid jet in another inviscid liquid

For the case of an inviscid liquid jet in another inviscid liquid, [35] becomes (Meister & Scheele 1967):

$$\beta^2 = \frac{2\eta(1-\eta^2)}{\dfrac{I_0(\eta)}{I_1(\eta)} + \dfrac{\hat{\rho}}{\rho}\dfrac{K_0(\eta)}{K_1(\eta)}}, \qquad [41]$$

which is identical to the result obtained by Christiansen (1955). Substituting $\mu = \hat{\mu} = 0$ into [37] yields

$$\left[1+\frac{1}{2}\frac{\hat{\rho}}{\rho}\eta\frac{K_0(\eta)}{K_1(\eta)}\right]\beta^2 = \eta^2(1-\eta^2), \qquad [42]$$

which may be rewritten as

$$\beta^2 = \frac{2\eta(1-\eta^2)}{\dfrac{2}{\eta} + \dfrac{\hat{\rho}}{\rho}\dfrac{K_0(\eta)}{K_1(\eta)}}. \qquad [43]$$

$I_0(\eta)/I_1(\eta) \sim 2/\eta$ for $\eta < 1$; therefore, [43] is essentially the same as [41]. From [39], [40] and [43], it can be seen that $\beta \to 0$ as $\hat{\rho}/\rho \to \infty$; i.e. the density ratio has a stabilizing effect. Equation [41] may be rewritten as

$$\beta^2 = \frac{2\eta(1-\eta^2)\dfrac{I_1(\eta)}{I_0(\eta)}}{1+\dfrac{\hat{\rho}}{\rho}\dfrac{I_1(\eta)}{I_0(\eta)}\dfrac{K_0(\eta)}{K_1(\eta)}}. \qquad [44]$$

Considering that

$$I_0(\eta)K_1(\eta) + I_1(\eta)K_0(\eta) = \frac{1}{\eta},$$

then

$$\frac{I_1(\eta)K_0(\eta)}{I_0(\eta)K_1(\eta)} = \frac{1}{\eta I_0(\eta)K_1(\eta)} - 1.$$

Since $I_0(\eta) \sim 1$ and $K_1(\eta) \sim 1/\eta$ for $\eta < 1$,

$$\frac{I_1(\eta)K_0(\eta)}{I_0(\eta)K_1(\eta)} \rightarrow 0.$$

Hence, [44] is equivalent to [36], i.e. the density effect may be neglected if the density ratio is of the order of 1. For this case the most unstable wavenumber is approximately $\eta_m = 0.697$.

Case 4: a highly viscous liquid jet in a gas

For the case of a highly viscous liquid jet in a gas, [35] reduces to (Meister & Scheele 1967):

$$\beta^2 + 6Z\eta^2\beta = \eta^2(1 - \eta^2), \quad [45]$$

which is identical to the classical result obtained by Weber (1931). Weber determined the most unstable wavenumber in this case to be $\eta_m = (2 + 6Z)^{-1/2}$. Substituting the appropriate conditions for this case, $\hat{\rho}/\rho \rightarrow 0$ and $\hat{\mu}/\mu \rightarrow 0$, into [34] yields the classical result [45].

Case 5: a highly viscous liquid jet in a low viscosity liquid

Meister & Scheele (1967) suggested that density ratio may be neglected when $\hat{\rho}/\rho < 6$, whereby the dispersion equation for a highly viscous liquid jet in an ambient liquid with low viscosity becomes the same as for the previous case, [45]. In this case, $\hat{\mu}/\mu \rightarrow 0$, thus [34] becomes

$$\left[1 + \frac{\hat{\rho}}{\rho}\frac{\eta}{2}\frac{K_0(\eta)}{K_1(\eta)}\right]\beta^2 + 6Z\eta^2\beta = \eta^2(1 - \eta^2). \quad [46]$$

If $\eta < 1$, then $K_0(\eta)/K_1(\eta) < 1$ and $\eta K_0(\eta)/K_1(\eta) \ll 1$. Thus, if $\hat{\rho}/\rho < 2$, the density ratio can be neglected and [46] reduces to [45]. As in the previous case, $\eta_m = (2 + 6Z)^{-1/2}$. Meister (1966) and Meister & Scheele (1967) did not provide details in their analyses on the effect of the density ratio. Numerical calculations performed in this study suggest that, to contain the resultant error, $\hat{\rho}/\rho$ should be < 2. Density ratios in the range $2 < \hat{\rho}/\rho < 6$ may result in large errors when utilizing [45] if $\eta > 0.5$.

Case 6: a low viscosity liquid jet in a highly viscous liquid

For the case of a liquid jet with low viscosity in a highly viscous liquid, [35] reduces to (Meister & Scheele 1967):

$$\beta^2 + 2\hat{Z}\eta^2\beta = \eta^2(1 - \eta^2), \quad [47]$$

where $\hat{Z} = \hat{\mu}(2a\rho\sigma)^{-1/2}$. Since in this case, $\hat{\mu}/\mu \gg 1$, the general solution developed in this study, [34], becomes

$$\left[1 + \frac{\hat{\rho}}{\rho}\frac{\eta}{2}\frac{K_0(\eta)}{K_1(\eta)}\right]\beta^2 + 2Z\frac{\hat{\mu}}{\mu}\eta^2\beta = \eta^2(1 - \eta^2). \quad [48]$$

Considering that $Z\hat{\mu}/\mu = \hat{Z}$ and $\eta K_0(\eta)/K_1(\eta) \ll 1$, if $\hat{\rho}/\rho < 2$, [48] reduces to [47]. Meister & Scheele (1967) determined the most unstable wavenumber for this case to be $\eta_m = (2 + 2\hat{Z})^{-1/2}$. From [34], [45], [47] and [48], it can be seen that $\beta \rightarrow 0$ as $Z \rightarrow \infty$ or $\hat{\mu}/\mu \rightarrow \infty$; i.e. both the viscosity ratio and the Ohnesorge number have stabilizing effects.

Case 7: a highly viscous liquid jet in another highly viscous liquid

The case of a highly viscous liquid jet in another highly viscous liquid was analyzed by Tomotika (1935). Tomotika derived the following relationship for this case:

$$Z\beta = (1 - \eta^2)F, \quad [49]$$

Table 1. The $\eta_m = \eta_m(\hat{\mu}/\mu)$ relation obtained by Tomotika (1935)

$\hat{\mu}/\mu$	0	0.2	1	1.1	2	10	100	∞
η_m	0	0.486	0.563	0.568	0.584	0.573	0.415	0

where $F = F(\eta, \hat{\mu}/\mu)$. Using numerical techniques, Tomotika calculated the most unstable wavenumber as a function of the viscosity ratio. Selected results of Tomotika's analysis are presented in table 1. Tomotika's results in table 1 are suspect:

(1) For $\hat{\mu}/\mu \ll 1$, the viscosity of the ambient fluid becomes negligible; hence, case 7 should reduce to case 5, and $\eta_m = (2 + 6Z)^{-1/2}$.
(2) For $\hat{\mu}/\mu \gg 1$, the viscosity of the jet fluid becomes negligible; hence, case 7 should reduce to case 6, and $\eta_m = (2 + 2\hat{Z})^{-1/2}$.
(3) Tomotika argued that the results presented in table 1 are validated by the apparent agreement with an experiment performed by Taylor which determined that $\eta_m = 0.5$ for $\hat{\mu}/\mu = 1.1$ (Tomotika 1935). It should be observed, however, that the most unstable wavenumbers calculated by Tomotika are relatively insensitive to the viscosity ratio over the range $0.2 < \hat{\mu}/\mu < 100$, all values being roughly 0.5. Hence, comparison with the single data point is inadequate to validate Tomotika's analysis.

The results of the present analysis yield the appropriate results at limiting viscosity ratios. For liquid-into-liquid systems, [34] reduces to

$$\beta^2 + 2Z\left(3 + \frac{\hat{\mu}}{\mu}\right)\eta^2\beta = \eta^2(1 - \eta^2). \qquad [50]$$

Based on Rayleigh's maximum instability theory, the most unstable wavenumber is obtained by applying the condition $\mathrm{d}\beta/\mathrm{d}\eta|_{\eta=\eta_m} = 0$ to [50]:

$$\eta_m = \left(2 + 6Z + 2Z\frac{\hat{\mu}}{\mu}\right)^{-1/2}. \qquad [51]$$

If $\hat{\mu}/\mu \ll 1$, case 7 reduces to case 5, and [51] yields Weber's classical result, $\eta_m = (2 + 6Z)^{-1/2}$. If $\hat{\mu}/\mu \gg 1$, case 7 reduces to case 6, and [51] yields

$$\eta_m = \left(2 + 2Z\frac{\hat{\mu}}{\mu}\right)^{-1/2} = (2 + 2\hat{Z})^{-1/2},$$

which matches Meister & Scheele's (1967) result for case 6. Table 2 compares selected results from this study, viz. [51], with numerical solutions to Tomotika's equation, [35], obtained by Kitamura *et al.* (1982). The parameters for the numerical solutions span the range $0.68 \leqslant \hat{\rho}/\rho \leqslant 1.47$, $0.06 \leqslant \hat{\mu}/\mu \leqslant 4.33$ and $1.96 \times 10^{-3} \leqslant Z \leqslant 9.96 \times 10^{-3}$, and thus represent non-limiting as well as limiting conditions. The most unstable wavenumbers predicted by Tomotika's limiting solution (Tomotika 1935) are also presented in table 2 for comparison.

In table 2, the most unstable wavenumbers predicted in this study agree well with the numerical solutions to [35]; by contrast, Tomotika's limiting solution substantially underestimates the most

Table 2. Most unstable wavenumber η_m predictions

$\hat{\mu}/\mu$	Numerical solution to [35][a]	This study, [51][b]	Tomotika's limiting solution[c]
0.06	0.611	0.623	0.353
0.10	0.643	0.688	0.409
0.66	0.659	0.682	0.530
1.15	0.643	0.682	0.568
2.56	0.666	0.702	0.583
4.33	0.651	0.693	0.581

[a]Obtained by Kitamura *et al.* (1982).
[b]The Z values used in [51] are the same as those used in the corresponding numerical solutions.
[c]Some values listed were obtained by interpolation (Tomotika 1935).

unstable wavenumber. It is believed that the large disagreement is due to Tomotika's neglect of the effect of the Ohnesorge number. Indeed, the Ohnesorge number and viscosity ratio are not independent. Under the condition, $\hat{\rho}/\rho < 2$, all limiting cases of liquid jets can be expressed by

$$\beta^2 + 2Z^*\eta^2\beta = \eta^2(1 - \eta^2), \qquad [52]$$

where $Z^* = (3\mu + \hat{\mu})/(2a\rho\sigma)^{1/2}$ is a modified Ohnesorge number. The corresponding most unstable wavenumber is $\eta_m = (2 + 2Z^*)^{-1/2}$. Thus, under such conditions, the most unstable wavenumber is influenced only by the modified Ohnesorge number.

5. CONCLUSIONS

For jets in ambient fluids conforming to Stokes flow, an explicit dispersion equation is obtained by employing an integro-differential approach. The dispersion equation of this study enables much easier prediction of the most unstable wavenumber and related information than Tomotika's implicit, complex dispersion equation. The equation developed here reduces to all limiting solutions to Tomotika's equation described in the literature.

Based on this study, the following are concluded:

(1) The effect of the viscosity ratio is influenced by the Ohnesorge number. In cases involving very large or very small viscosity ratios, the Ohnesorge number becomes the only significant parameter in the dispersion equation. Tomotika's prediction relating to the effect of the viscosity ratio does not approach the appropriate limiting solutions and, therefore, is suspect.

(2) For the conditions considered in this study, the effect of the density ratio is weaker than those of the viscosity ratio or Ohnesorge number, and can be neglected in liquid-into-gas and many liquid-into-liquid systems; however, in gas-into-liquid systems, the density ratio could be a dominant factor.

(3) All three parameters—density ratio, viscosity ratio and Ohnesorge number—tend to stabilize the jet.

(4) For all limiting cases involving liquid jets, if $\hat{\rho}/\rho < 2$, the most unstable wavenumber can be expressed by the general equation, $\eta_m = (2 + 2Z^*)^{-1/2}$.

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